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ON COMBINATIONS OF RANDOM LOADS

by

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and

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Structures are subject to changing loads from various sources. In many instances these loads fluctuate in time in an apparently random fashion. Certain loads vary rather slowly (called constant loads); other loads occur more nearly as impulses (shock loads). Suppose that the stress put on the structure by various loads acting simultaneously can be expressed as a linear combination of the load magnitudes. In this paper certain simple but somewhat realistic probabilistic load models are given and the resulting probabilistic

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20. ABSTRACT CONTINUED

model of the total stress on the structure caused by the loads is considered. The distribution of the first time until the stress on the structure exceeds a given level x, and the distribution of the maximum stress put on the structure during the time interval $(0,t]$ are studied. Asymptotic properties are also given. It is shown that the asymptotic properties of the maximum stress are related to those of the maxima of a sequence of dependent random variables. Classical extreme value type results are derived under proper normalization.

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ON COMBINATIONS OF RANDOM LOADS

by

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and

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1. Introduction and Assumptions

The integrity of a great many physical structures is potentially threatened by combinations of physical loads of varying magnitudes from various sources. We are thinking of structures such as buildings (for instance those that house or contain nuclear power plant elements); aircraft and spacecraft; electrical transmission networks; bridges, piers, and dams; and offshore oil drilling rigs that experience loads from wind, snow and ice, tides, earthquakes, and so forth. In many instances the total load or stress experienced by a structure varies in time in an apparently random fashion. Certain load components vary rather slowly; for example, that component resulting from snow and ice accumulation; others occur more nearly as impulses, such as those associated with winds or earthquakes. The problem is to design structures to withstand the superposition of loads from many sources with at least an approximately understood (high) probability. In engineering terms we wish to work towards

developing a rational safety factor technique for designing structures to withstand the combination of loads anticipated. The purpose of this paper is to describe and investigate certain simple but somewhat realistic probabilistic load models for use in design, and perhaps safety, assessment of structures.

In this paper we confine attention to the superposition of just two load types: <u>shock loads</u>, and <u>constant loads</u>. For example, wind gusts, flash floods, and earthquakes have varying magnitudes and have relatively short durations in comparison to the times between their occurrences; these will be modeled as instantaneous shock loads. On the other hand snow, ice, or water accumulation, or even the presence of slow-moving vehicles or furniture, present loads that remain nearly constant in time, occasionally changing to new levels; these will be modeled as constant loads that change infrequently. Throughout this investigation it will be assumed that the effective stress exerted by several types of loads acting simultaneously can be expressed as a linear combination (actually, sum) of those component loads, the loads being treated as stochastic processes. Somewhat similar load combination models have been studied in the past; see the work of Bosshardt [1975], Larrabee and Cornell [1978], McGuire and Cornell [1974], Pier and Cornell [1973], Weh [1977], and others. Note that the present study utilizes the notion of stochastic point processes as a modeling tool; see Cox and Miller [1965] for an introduction, and Lewis (ed.) [1972] for a more extensive treatment.

1.1. Modeling Assumptions and Problems

Assume that the successive magnitudes of shock loads are independently and identically distributed (i.i.d.) random variables with common continuous distribution function $G(y)$; $G(y) = 0$ for $y \leq 0$. The shock loads' appearance is regulated by a Poisson process with rate μ. Furthermore, the constant loads have i.i.d. magnitudes with distribution $F(x)$; $F(x) = 0$ for $x \leq 0$. Constant loads change at moments of a Poisson process with rate λ. Finally, the shock and constant load processes are, for the most part, taken to be statistically independent, although this assumption can be relaxed at times without difficulty.

Let $X(t)$ (respectively $Y(t)$) be the magnitude of the constant (shock) load process at time t; ($Y(t)$ will usually be zero). Put $Z(t) = X(t) + Y(t)$, the superposition of the two loads at time t, and $M(t) = \sup_{s \leq t} Z(s)$, the maximum load combination in $[0,t]$. See Figure 1. Let $T_x = \{\inf t \geq 0 : Z(t) > x\}$,

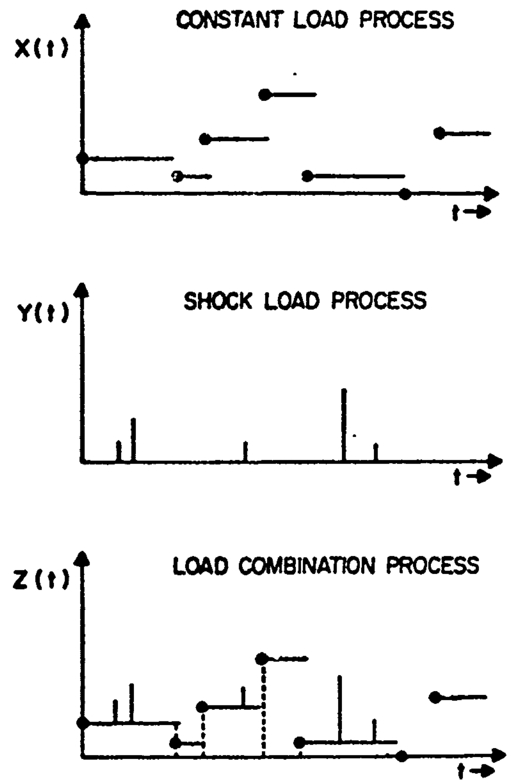

Fig. 1
THE LOAD COMBINATION

i.e. the first-passge time to a combined stress level x. Unless fatigue or some other cumulative effect is in operation T_x will represent the time to failure of a structure whose strength is x, and that is subjected to a stress history $\{Z(t),\ t \geq 0\}$.

Section 2 is concerned with the description of the distribution of the maximum process and the distribution of T_x. The first step is to obtain Laplace transforms with respect to time. Asymptotic results $(x \to \infty)$ for $E[T_x]$ and the distribution of the normalized value $T_x^* = T_x/(E[T_x])$ will then be presented. In particular it will be shown that the distribution of T_x^* is approximately a unit exponential distribution as x becomes large and so does $E[T_x]$.

Section 3 is devoted to a study of the asymptotic properties of the "maximum process," $M(t)$, as $t \to \infty$. These results relate to those of Welsch [1972] and O'Brien [1974a,b]; they are seen to extend the classical "extreme value" results of Gumbel [1958] and Gnedenko [1943]. We present results for the asymptotic distribution of $M(t)$ and for the joint distribution of the first and second maximum load combination in $[0,t]$ as $t \to \infty$.

2. The Laplace Transform of the Distribution of $M(t)$ and a First-Passage Time Limit Theorem

2.1. Towards the Distribution of $M(t)$

Consider first the distribution function of the maximum process, $H_x(t) = P\{M(t) \le x\}$, for $x > 0$. Its Laplace transform is available immediately by noting that if T_1 is the time of the first change in the magnitude of the constant load process, then

$$
\begin{aligned}
H_x(t) &= P\{M(t) \le x,\ T_1 > t\} + P\{M(t) \le x,\ T_1 \le t\} \\
&= e^{-\lambda t} \int_0^x \exp\{-\mu t\, \bar{G}(x-y)\}\, F(dy) \\
&\quad + \int_0^t \lambda e^{-\lambda v}\, dv \int_0^x \exp\{-\mu v\, \bar{G}(x-y)\}\, H_x(t-v)\, F(dy)\,. \qquad (2.1)
\end{aligned}
$$

Next take Laplace transforms with respect to t:

$$
\hat{h}_x(\xi) \equiv \int_0^\infty e^{-\xi t} H_x(t)\,dt = M_x(\xi) + \lambda M_x(\xi)\, \hat{h}_x(\xi)\ ; \qquad (2.2)
$$

reversal of the order of integration provides that

$$
M_x(\xi) = \int_0^x [\xi + \lambda + \mu \bar{G}(x-y)]^{-1}\, F(dy)\,. \qquad (2.3)
$$

Hence,

$$\hat{h}_x(\xi) = \frac{M_x(\xi)}{1 - \lambda M_x(\xi)} . \tag{2.4}$$

It seems to be difficult to solve (2.1) (invert (2.4)) in any simple convenient form for any interesting choice of the distributions F and G. However, useful information can still be gleaned from (2.4). First, it is clear that if T_x is the first-passage time to combined load x, so

$$T_x = \inf\{t \geq 0 : M(t) > x\} , \tag{2.5}$$

then, since $P\{M(t) \leq x\} = P\{T_x > t\}$,

$$h_x(\xi) = \int_0^\infty e^{-\xi t} P\{T_x > t\} dt . \tag{2.6}$$

Let $\xi \to 0$ in (2.4) to find that

$$m(x) \equiv E[T_x] = h_x(0) = \frac{M_x(0)}{1 - \lambda M_x(0)} . \tag{2.7}$$

We now record some expressions for the mean first-passage time to x when specific distributions for shock and constant load magnitudes are in force.

<u>Example 2.1</u>. Identical exponentials: $F(x) = G(x) = 1 - e^{-x}$, $x \geq 0$. In this case

$$M_x(0) = \frac{1}{\lambda}(1 - e^{-x}) - \frac{\mu x e^{-x}}{\lambda^2} - \frac{\mu e^{-x}}{\lambda} \ln\left[\frac{\lambda + \mu e^{-x}}{\lambda + \mu}\right]$$

and

$$E[T_x] = \frac{e^x}{\lambda^2}\left\{\frac{\lambda(1-e^{-x}) - \mu x e^{-x} + -\mu e^{-x} \ln[(\lambda+\mu)/(\lambda+\mu e^{-x})]}{1 + \frac{\mu}{\lambda} x + \frac{\mu}{\lambda} \ln[(\lambda+\mu)/(\lambda+\mu e^{-x})]}\right\} \tag{2.8}$$

$$\sim \frac{1}{\mu}\left(\frac{e^x}{x}\right) \qquad \text{as} \quad x \to \infty .$$

<u>Example 2.2</u>. Different exponentials:

$$\bar{F}(x) = e^{-ax}, \qquad \bar{G}(x) = e^{-kx} \qquad \text{where } \frac{a}{k} = 2.$$

In this case

$$M_x(0) = \int_0^x [\lambda + \mu e^{-k(x-y)}]^{-1} a e^{-ay}\, dy$$

$$= \lambda^{-2} e^{-2kx} N(x)$$

where

$$N(x) = \lambda[-1 + e^{2kx}] + 2\mu[1-e^{kx}]$$

$$+ 2\mu^2[-kx + \ln[(\lambda+\mu)(\lambda + \mu e^{-kx})^{-1}]].$$

Thus

$$E[T_x] = \lambda^{-1} N(x) \; [\lambda e^{2kx} - N(x)]^{-1} \sim \frac{e^{kx}}{2\mu} . \qquad (2.9)$$

Example 2.3. $\bar{F}(x) = e^{-x}$, $\bar{G}(x) = e^{-x^2}$. In this case

$$M_x(0) = \frac{1}{\lambda} [1 - e^{-x}] + \frac{\mu}{\lambda} e^{-x} N(x)$$

where

$$N(x) = (e^{-x} - 1)[(\lambda + \mu)(\lambda + \mu e^{-x})]^{-1} .$$

Thus

$$E[T_x] = \frac{1}{\lambda} e^{x} \; [1 - e^{-x} + \mu e^{-x} N(x)][1 - \mu N(x)]^{-1} \sim \frac{\lambda + \mu}{\lambda(\lambda+\mu) + \mu} e^{x}. \qquad (2.10)$$

Example 2.4.

$$\bar{F}(x) = \begin{cases} 1 & x < a_1^{1/\alpha} , \\ a_1 x^{-\alpha} & x \geq a_1^{1/\alpha} , \end{cases}$$

and

$$\bar{G}(x) = \begin{cases} 1 & x < a_2^{1/\alpha} , \\ a_2 x^{-\alpha} & x > a_2^{1/\alpha} \end{cases}$$

where $\alpha > 0$ and $a_1, a_2 > 0$. In this case as $x \to \infty$

$$E[T_x] \sim [\lambda a_1 + \mu a_2]^{-1} x^{\alpha}. \qquad (2.11)$$

Example 2.5.

$$\bar{F}(x) = \begin{cases} 1 & \text{if} \quad x < 1 , \\ x^{-\alpha} & \text{if} \quad x \geq 1 , \end{cases}$$

and

$$\bar{G}(x) = \begin{cases} 1 & \text{if} \quad x < 1 , \\ x^{-\beta} & \text{if} \quad x \geq 1 . \end{cases}$$

If $\alpha < \beta$, then as $x \to \infty$

$$E[T_x] \sim \frac{1}{\lambda} x^{\alpha} .$$

If $\beta < \alpha$, then as $x \to \infty$ (2.12)

$$E[T_x] \sim \frac{1}{\mu} x^{\beta} .$$

The next result is a limiting result for the first time the load combination process exceeds a given level x.

THEOREM (2.1). The limiting distribution of T_x is exponential, in the sense that

$$\lim_{x \to x_0} P\{m(x)^{-1} T_x > t\} = e^{-t}$$

where

$$x_0 = \inf\{t : F*G(t) = 1\}.$$

and

$$m(x) = E[T_x].$$

Proof. Put $\rho = \mu\lambda^{-1}$ and rearrange (2.3) so that the denominator of (2.7) is in the form

$$1 - \lambda M_x(0) = \bar{F}(x) + \int_0^x \rho\bar{G}(x-y)[1 + \rho\bar{G}(x-y)]^{-1} F(dy) \equiv h_1(x). \quad (2.13)$$

Now apply the bounded convergence theorem and the fact that F and G have densities to show that $m(x) \to \infty$ as $x \to x_0$. It next follows by a change of variables in (2.4) that

$$\int_0^\infty e^{-\xi t} P\{m(x)^{-1} T_x > t\}dt = m(x)^{-1} \hat{h}_x(\gamma(x)) = \frac{M_x(\gamma(x))}{m(x)[1-M_x(\gamma(x))]} \quad (2.14)$$

where $\gamma(x) = \xi m(x)^{-1}$. By the bounded convergence theorem applied to (2.3),

$$\lim_{x \to x_0} M_x(\gamma(x)) = \frac{1}{\lambda} . \quad (2.15)$$

Again from (2.3) we have

$$1 - \lambda M_x(\gamma(x))$$

$$= \bar{F}(x) + \int_0^x [\gamma(x)\lambda^{-1} + \rho\bar{G}(x-y)][\gamma(x)\lambda^{-1} + 1 + \rho\bar{G}(x-y)]^{-1} F(dy) .$$

Hence,

$$m(x)\,[1 - \lambda M_x(\gamma(x))]$$

$$= \frac{\xi}{\lambda} \int_0^x [\gamma(x)\lambda^{-1} + 1 + \rho\bar{G}(x-y)]^{-1}\, F(dy)$$

$$+ m(x)\,[\bar{F}(x) + \int_0^x \rho\bar{G}(x-y)\,[\gamma(x)\lambda^{-1} + 1 + \rho\bar{G}(x-y)]^{-1}\, F(dy)]\ .$$

Let

$$h_2(x) = \bar{F}(x) + \int_0^x \rho\bar{G}(x-y)\,[\gamma(x)\lambda^{-1} + 1 + \rho\bar{G}(x-y)]^{-1}\, F(dy)\ ;$$

then

$$h_1(x) - h_2(x)$$

$$= \int_0^x F(dy)\ \rho\bar{G}(x-y)\{-[1 + \rho\bar{G}(x-y)] + [\gamma(x)\lambda^{-1} + 1 + \rho\bar{G}(x-y)]\}$$

$$\times\{[1 + \rho\bar{G}(x-y)]\ [\gamma(x)\lambda^{-1} + 1 + \rho\bar{G}(x-y)]\}^{-1}.$$

$$= \int_0^x F(dy)\ \rho\bar{G}(x-y)\ \gamma(x)\lambda^{-1}\ \{[1 + \rho\bar{G}(x-y)]\ [\gamma(x)\lambda^{-1} + 1 + \bar{G}(x-y)]\}^{-1}.$$

$$\equiv \gamma(x)\lambda^{-1}\ k(x)$$

where

$$k(x) = \int_0^x \rho\bar{G}(x-y)\{[1 + \rho\bar{G}(x-y)][\gamma(x)\lambda^{-1} + 1 + \rho\bar{G}(x-y)]\}^{-1} F(dy).$$

By the bounded convergence theorem

$$\lim_{x \to x_0} k(x) = 0 .$$

Recall that

$$m(x) = \frac{\frac{1}{\lambda} + o(x)}{h_1(x)} .$$

Hence,

$$\frac{h_1(x) - h_2(x)}{m(x)} = \frac{\xi h_1(x)\lambda^{-1}}{\lambda^{-1} + o(x)} \frac{k(x)}{h_1(x)}$$

$$= \frac{\xi}{1 + o(x)} k(x)$$

which tends to zero as $x \to x_0$. Therefore,

$$\lim_{x \to x_0} m(x)[1 - \lambda M_x(\gamma(x))] = \xi\lambda^{-1} + \lambda^{-1} \lim_{x \to x_0} \left[1 + \frac{h_2(x) - h_1(x)}{h_1(x)}\right]$$

$$= (\xi + 1) \cdot \lambda^{-1}.$$

The result now follows from (2.14), (2.15) and the unicity result for Laplace transforms.

3. Asymptotic Results for the Maximum Load Combination

In this section we will study the asymptotic behavior of the maximum load combination to occur during time interval $(0,t]$ as $t \to \infty$. Most results concerning maxima of non-Gaussian random variables are for variables in discrete time. Thus we will first obtain results for an embedded discrete time load combination process.

Let S_n be the time of the nth change in magnitude of the load combination process; that is,

$$S_1 = \inf\{t > 0 : Z(t) \neq Z(t-)\}$$

and for $n > 1$ (3.1)

$$S_n = \inf\{t > S_{n-1} : Z(t) \neq Z(t-)\}.$$

The change in the load combination at time S_n may be due to either a change in the constant load process or to the arrival of a shock load. Let $Z_n = Z(S_n)$ (respectively, $X_n = X(S_n)$, and $Y_n = Y(S_n)$) be the magnitude of the load combination (respectively constant load and shock load) at time S_n. Put

$$M_n = \max_{0 \leq k \leq n} (Z_k) . \tag{3.2}$$

Note that the times $\{S_n\}$ are the arrival times of a Poisson process with rate $\lambda + \mu$ and are independent of $\{Z_n\}$. Further,

$$P\{Z_0 \le x\} = F(x)$$

and for $n \ge 1$

$$P\{Z_n \le x\} = pF(x) + qF*G(x) \tag{3.3}$$

where

$$p = \int_0^\infty \lambda e^{-\lambda t} e^{-\mu t} dt = \frac{\lambda}{\lambda + \mu}$$

and

$$q = 1-p.$$

3.1. Properties of An Imbedded Maximum Process

We will first study the asymptotic behavior of M_n as $n \to \infty$. Note that the random variables $\{Z_n\}$ are not independent. In fact $\{(X_n, Z_n); n = 0,1,\ldots\}$ is a discrete time Markov process. The following result describes the dependence of the random variables $\{Z_n\}$.

PROPOSITION (3.1). The sequence $\{Z_n\}$ is uniformly mixing (cf. Loynes [1965, p. 994]).

Proof. Let A and B be events such that $A = f(Z_0,\ldots,Z_n)$ and $B = g(Z_{n+m}, Z_{n+m+1}, \ldots)$. Then

$$\begin{aligned}
&|P(A \cap B) - P(A)\,P(B)| \\
&\quad = |E[(P(B|X_n) - P(B)); A]| \\
&\quad = q^m |E[\textstyle\int(\varepsilon_{X_n}(dy) - F(dy))\,P(B|X_{n+m} = y); A]| \\
&\quad \le 2q^m\,P(A)
\end{aligned}$$

where ε_x denotes the Dirac measure concentrated at the point x.

This last proposition and a result of Loynes [1965, Theorem 1] imply that the only possible nondegenerate limiting distributions of M_n are the same three types that occur in the asymptotic behavior of the maxima of independent random variables; namely (except for scale and location parameters),

$$H_1(x) = \begin{cases} 0 & x \le 0 , \\ \exp\{-(x^{-\alpha})\} & x > 0, \quad \alpha > 0 \end{cases}$$

$$H_2(x) = \begin{cases} \exp\{-(-x^{\alpha})\} & x < 0, \quad \alpha > 0 \\ 1 & x \ge 0 \end{cases} \tag{3.4}$$

$$H_3(x) = \exp\{-e^{-x}\} \qquad -\infty < x < \infty .$$

However, for the process $\{Z_n\}$, the following Proposition (3.2) together with a result of Loynes [1965, Lemma 2] implies that the limiting behavior of M_n will not necessarily be the same as that of the maximum of a sequence of <u>independent</u> random variables having common distribution $pF(x) + qF{*}G(x)$.

PROPOSITION (3.2). For any sequences $k_n \to \infty$ and $c_n \to \infty$

$$\lim_{n \to \infty} \frac{1}{k_n} \sum_{i=1}^{k_n} (k_n - i)\ P\{Z_{i+1} > c_n | Z_1 > c_n\} > 0 .$$

Proof. Note that for any $c > 0$

$$P\{Z_{i+1} > c | Z_1 > c\} \geq q^i .$$

The result now follows.

In order to obtain the limiting distribution of M_n we will first study the limiting distribution of a related process which is defined as follows.

Let

$$J_n = \begin{cases} 1 & \text{if } Y_n = 0 , \\ 0 & \text{if } Y_n > 0 ; \end{cases}$$

that is, $J_n = 1$ if the nth change in the load combination process $\{Z(t); t \geq 0\}$ is due to a change in the constant load and 0 if it is due to the arrival of a shock load.

Let

$$\tau_0 = 0$$

and

$$\tau_n = \inf\{k > \tau_{n-1} : J_k = 1\} ,$$

the index of the change in the load combination process which is due to the nth change in the constant load process. Let

$$L_n(k) = \sum_{\tau_i \leq n} 1_{\{k\}}(\tau_i - \tau_{i-1}) ,$$

the number of τ_i that occur before (discrete) time n such that $\tau_i - \tau_{i-1} = k$; ($1_{\{k\}}(x) = 1$ if $x = k$ and 0 otherwise).

Let

$$U_n = \sup_{\tau_{n-1} \leq k < \tau_n} Z_k ,$$

the maximum load combination to occur during the nth period of time the constant load process remains constant.

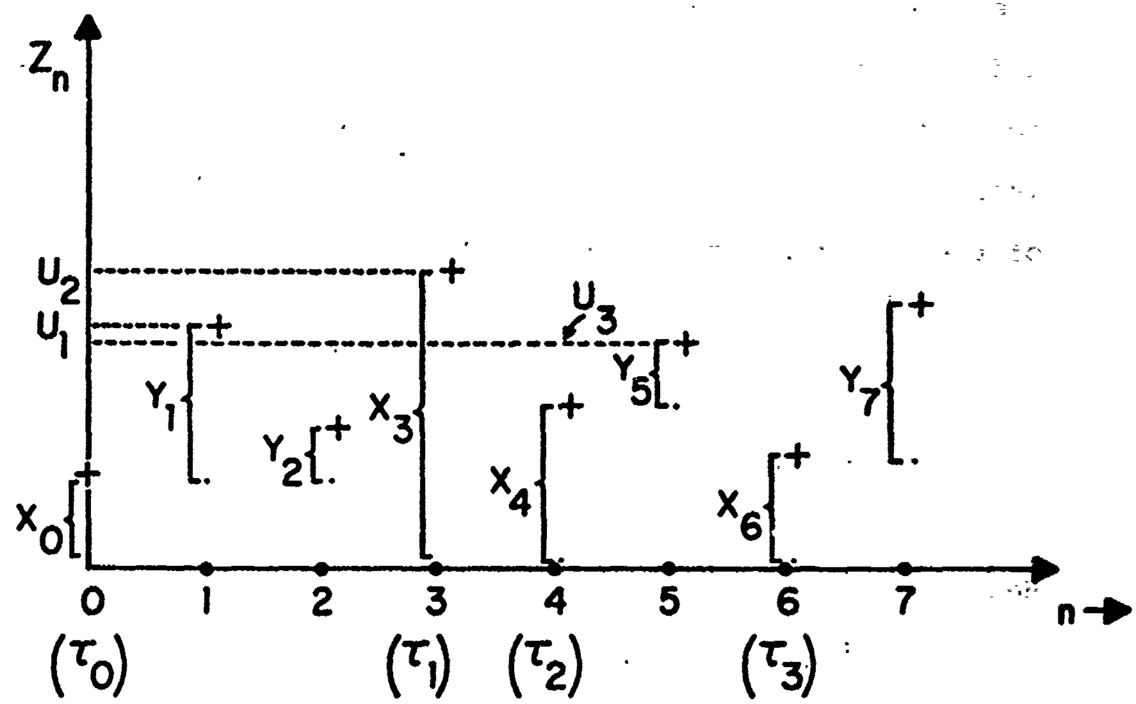

Fig. 2

THE DISCRETE TIME LOAD COMBINATION PROCESS AND MAXIMA.

For each integer K

$$M_{K,n} = \max_{\tau_i \le n} U_i \, 1_{[0,K]}(\tau_i - \tau_{i-1}) ,$$

the maximum load combination for the discrete time process in the time interval $[0,n]$ which is due to those shock loads and constant loads which occur during time intervals of constancy for the discrete time constant load process which are of length less than or equal to K. Finally, let $M_{K,n}^{(2)}$ be the magnitude of the second maximum of the sequence

$$\{U_i \, 1_{[0,K]}(\tau_i - \tau_{i-1}); \tau_i \le n\} .$$

Note from (3.2) that

$$M_{K,n} \le M_n .$$

Hence

$$\begin{aligned} H_{K,n}(x) &\equiv P\{M_{K,n} \le x\} \\ &= E[\prod_{k=0}^{K} [F*G^k(x)]^{L_n(k)}] \\ &\ge P\{M_n \le x\} . \end{aligned}$$

3.2. Limiting Results for Pareto-tail Load Distributions

In what follows we will assume

$$\bar{F}(x) = a_1 x^{-\alpha} L(x), \qquad x > 0,$$
$$\bar{G}(x) = a_2 x^{-\alpha} L(x), \qquad x > 0, \tag{3.5}$$

where L is a slowly varying function.

Next we will find the limiting distribution of $M_{K,n}$ as $n \to \infty$ for fixed K. Then we will show that the limiting distribution of $M_{K,n}$ is close to that of M_n for K large.

PROPOSITION (3.3). Let $\bar{F}$ and $\bar{G}$ be as in (3.5), and u_n be such that

$$(u_n)^{-\alpha} L(u_n) \sim \frac{1}{n} \quad \text{as} \quad n \to \infty. \tag{3.6}$$

Then

$$\lim_{n\to\infty} H_{K,n}(u_n x) = \exp\{-x^{-\alpha} p^2 \sum_{k=0}^{K} q^k [a_1 + ka_2]\} \equiv H_k(x).$$

Proof. Note that

$$\begin{aligned} 1 - G^k(x) &= k\bar{G}(x)\, G(x)^{k-1} \\ &\quad + \binom{k}{2} \bar{G}(x)^2\, G(x)^{k-2} \\ &\quad + \cdots + [\bar{G}(x)]^k \\ &\sim ka_2 x^{-\alpha} L(x) \qquad \text{as} \qquad x \to \infty. \end{aligned}$$

Thus by result (8.13) on page 278 of Feller [1971],

$$1 - F{*}G^k(x) \sim (a_1 + ka_2)x^{-\alpha} L(x) \quad .$$

Hence as $n \to \infty$

$$H_{K,n}(u_n x) = E[\exp\{\sum_{k=0}^{K} L_n(k) \ \ell n \ F{*}G^k(u_n x)\}]$$

$$\sim E[\exp\{\sum_{k=0}^{K} L_n(k) [1 - F{*}G^k(u_n x)\}]$$

$$\sim E\left[\exp\left\{\sum_{k=0}^{K} \frac{L_n(k)}{n} x^{-\alpha}[a_1 + ka_2]\right\}\right] .$$

Note that

$$\lim_{n \to \infty} \frac{L_n(k)}{n} = p^2 q^k \quad , \qquad k = 0, 1, \ldots \ .$$

The result now follows by the bounded convergence theorem.

THEOREM (3.4). Under the assumptions of Proposition (3.3)

$$\lim_{n \to \infty} P\{M_n \leq u_n x\} = \exp\{-x^{-\alpha}[pa_1 + qa_2]\} \ .$$

<u>Proof</u>. Let

$$H_K(x) = \exp\{\sum_{k=0}^{K} p^2 q^k x^{-\alpha}[a_1 + ka_2]\}$$

and

$$H(x) = \lim_{K \to \infty} H_K(x)$$

$$= \exp\{-x^{-\alpha}[pa_1 + qa_2]\} .$$

$$|P\{M_n \le u_n x\} - H(x)|$$

$$\le |P\{M_n \le u_n x\} - H_{K,n}(u_n x)|$$

$$+ |H_{K,n}(u_n x) - H_K(x)| + |H_K(x) - H(x)| .$$

Note that

$$|P\{M_n \le u_n x\} - H_{K,n}(u_n x)| \le E[1 - \exp\{-A_n(K)\}] . \tag{3.7}$$

where

$$A_n(K) = - \sum_{k=K+1}^{\infty} L_n(k)\ \ell n\ F{*}G^k(u_n x)$$

$$= \sum_{k=K+1}^{\infty} L_n(k)\ [1 - F{*}G^k(u_n x) + o(1 - F{*}G^k(u_n x))] .$$

Since

$$1 - G^k(x) \le k\bar{G}(x)$$

and

$$1 - F{*}G^k(x) = 1 - G^k(x) + k \int_0^x G^{k-1}(y)\ \bar{F}(x - y)\ G(dy)$$

$$\le k\ \overline{F{*}G}(x)$$

it follows that

$$A_n(K) \leq \sum_{k=K+1}^{\infty} L_n(k) [k\, \overline{F*G}(u_n x) + o(k\, \overline{F*G}(u_n x))] \ .$$

Note that, for $0 < \varepsilon < 1$,

$$\overline{F*G}(u_n x) \leq \bar{F}(u_n x(1-\varepsilon)) + \bar{G}(u_n x(1-\varepsilon)) + \bar{F}(u_n x\varepsilon)\ \bar{G}(u_n x\varepsilon) \ .$$

Therefore for fixed $\delta > 0$ there is an N_0 such that for $n \geq N_0$

$$\sum_{k=K+1}^{\infty} L_n(k)\ k\ \overline{F*G}(u_n x) \leq \sum_{k=K+1}^{\infty} \frac{L_n(k)}{n}\, k\, [a_1 x^{-\alpha} + a_2 x^{-\alpha} + \delta] \ .$$

Further,

$$E[L_n(k)] \leq E[\sum_{i=1}^{n} 1_{\{k\}}(\tau_i - \tau_{i-1})] = np^2 q^k \ .$$

Hence for $n \geq N_0(\delta)$

$$E[\sum_{k=K+1}^{\infty} L_n(k)\ k\ \overline{F*G}(u_n x)] \leq [a_1 + a_2]x^{-\alpha} + \delta] \sum_{k=K+1}^{\infty} p^2 q^k k. \qquad (3.8)$$

Choose $\gamma > 0$. Apply Jensen's inequality to (3.7): for $n > N_0(\delta)$

$$E[1 - \exp\{-A_n(K)\}] \leq 1 - \exp\{E[-A_n(K)]\} + 0$$

as $K \to \infty$. Thus for sufficiently large n and K

$$|P\{M_n \le u_n x\} - H_{K,n}(u_n x)|$$

is arbitrarily small. It follows that

$$\lim_{n \to \infty} P\{M_n \le xu_n\} = \exp\{-x^{-\alpha}[pa_1 + qa_2]\} \qquad (3.9)$$

whereas, if the load process were one of independent random variables, denoted by Z_n' with distribution $pF(x) + qF*G(x)$, the maximum would be distributed as follows

$$\lim_{n \to \infty} P\{M_n' \le xu_n\} = \exp\{-x^{-\alpha}[a_1 + qa_2]\} \; ; \qquad (3.10)$$

The u_n is the same in each case; see (3.6). Comparison of (3.9) and (3.10) shows that the expression (3.10) overestimates the probability of exceeding a given stress level.

O'Brien [1974a] obtained the above result in the case in which there is a constant load (i.e., $a_2 = 0$) alone, or only a shock load (i.e., $a_1 = 0$).

3.3. Results for Continuous Time

THEOREM (3.5). Let $\bar{F}$ and $\bar{G}$ be as in (3.5). Let $u(t)$ be such that $u(t)^{-\alpha} L(u(t)) \sim 1/t$ as $t \to \infty$. Then $\lambda + \mu$.

$$\lim_{t \to \infty} P\{M(t) \leq xu(t)\} = \exp\{-x^{-\alpha}[\lambda a_1 + \mu a_2]\} . \quad (3.9)$$

Proof. Let $N(t)$ be the number of changes in the load combination process, $\{Z(u); u \geq 0\}$, in the time interval $[0,t)$. The process $\{N(t); t \geq 0\}$ is a Poisson process with rate

Further,

$$M(t) = M_{N(t)} .$$

The result now follows from the proofs of Proposition (3.3) and Theorem (3.4), the strong law of large numbers for $N(t)$, and the bounded convergence theorem.

Slight modifications of the proofs of Proposition (3.3) and Theorems (3.4) and (3.5) imply the following result.

THEOREM (3.6), Let

$$\bar{F}(x) = x^{-\alpha}L(x) \quad \text{and} \quad \bar{G}(x) = x^{-\beta}L(x)$$

where L is a slowly varying function.

a) If $\alpha < \beta$ and $u(t)$ is such that $u(t)^{-\alpha}L[u(t)] \sim 1/t$ as $t \to \infty$, then

$$\lim_{t \to \infty} P\{M(t) \leq xu(t)\} = \exp\{-\lambda x^{-\alpha}\}. \quad (3.10)$$

b) If $\beta < \alpha$ and $u(t)$ is such that $u(t)^{-\beta} L[u(t)] \sim 1/t$ as $t \to \infty$, then

$$\lim_{t \to \infty} P\{M(t) \leq xu(t)\} = \exp\{-\mu x^{-\beta}\}.$$

The above result indicates that if the tail of the distribution of the magnitude of a constant load (respectively shock load) dominates that of the shock (respectively constant load), the asymptotic behavior of the maximum load combination is the same as that of the maximum constant load (respectively shock load) by itself. On the other hand, Theorem (3.5) indicates that if tails of the distributions of the magnitudes of the constant and shock loads are comparable, then the asymptotic behavior of the maxima of the load combination process depends on both the constant and shock load processes.

3.4. Limiting Results for the Joint Distribution of the First and Second Maxima

Let $M_n^{(2)}$ denote the magnitude of the second maxima of $\{Z_k; k \leq n\}$. Assume $\bar{F}$ and $\bar{G}$ are as in (3.5). Let

$$H(x) = \exp\{-x^{-\alpha}[pa_1 + qa_2]\}$$

as before. Techniques similar to those used in the proofs of the results in Subsection (3.2) yield the following result

THEOREM (3.7). Let u_n be as in (3.6). Then

$$\lim_{n \to \infty} P\{M_n^{(2)} \le yu_n, M_n \le xu_n\} = B(y,x)$$

where for $x \le y$,

$$B(y,x) = H(x)$$

and for $y < x$

$$B(y,x) = H(y) \{1 + [p^2 a_1 + qa_2][y^{-\alpha} - x^{-\alpha}]\} .$$

A result of Welsch [1972] implies that the limiting distribution B must be of the following form

$$B(y,x) = \begin{cases} H(x) & \text{if } y > x, \\ H(y)\{1 - g[(\ell n\ H(x)/\ell n\ H(y))]\ell n\ H(y)\} , & \text{if } y < x \end{cases} \tag{3.11}$$

where $g(s)$, $0 < s \le 1$ is a concave, nonincreasing function which satisfies $g(0)(1 - s) \le g(s) \le 1 - s$. In our case

$$g(s) = \left[\frac{p^2 a_1 + qa_2}{pa_1 + qa_2} \right] (1 - s) . \tag{3.12}$$

Let $M^{(2)}(t)$ denote the magnitude of the second maximum to occur in $[0,t]$ for the continuous time load combination process $\{Z(s); s \geq 0\}$. Arguments similar to those in Section 3.3 can be used to obtain the following result.

THEOREM (3.8). Let F and G be as in (3.5) and $u(t)$ be such that $u(t)^{-\alpha} L(u(t)) \sim 1/t$ as $t \to \infty$. For $x > y$

$$\lim_{t \to \infty} P\{M(t) \leq xu(t), M^{(2)}(t) \leq yu(t)\}$$

$$= \exp\{-y^{-\alpha}[\lambda a_1 + \mu a_2]\} \left[1 + \frac{\lambda^2}{\lambda+\mu} a_1 + \mu a_2 (y^{-\alpha} - x^{-\alpha})\right];$$

for $x \leq y$

$$\lim_{t \to \infty} P\{M(t) \leq xu(t), M^{(2)}(t) \leq yu(t)\}$$

$$= \exp\{-x^{-\alpha}[\lambda a_1 + \mu a_2]\}.$$

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